METRIC AND TOPOLOGICAL SPACES: EXAM 2025/26

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Problem 1 (10%). The discrete metric d_0 attains exactly two different real values. Does there exist a metric function $\varrho\colon \mathcal{X}\times\mathcal{X} \to \mathbb{R}$ which attains exactly 2025 different real values ?
Either prove the non-existence or give an explicit example $(\mathcal{X}, \varrho)$.

(a) **Problem 2** (15+15%). Find out whether in metric spaces $(\mathcal{X}, d_{\mathcal{X}})$, each "closed disk" $B_{\underline{q}}^{d_{\mathcal{X}}}(x_0) = \{x \in \mathcal{X} \mid d_{\mathcal{X}}(x, x_0) \leqslant q\}$ is closed.

(b) • Does each closed disk always coincide with the closure $\overline{B_q^{d_{\mathcal{X}}}(x_0)}$ of the open disk of radius q centered at $x_0 \in \mathcal{X}$? discrete metric ?
In items (a), (b) either prove "always" or give a counterexample.

Problem 3 (20%). Let $(\mathcal{X}, d_{\mathcal{X}})$ be a metric space and $\{U_i \mid i \in I\}$ be a family of connected subsets $U_i \subseteq \mathcal{X}$ such that $U_i \cap U_j \neq \varnothing$ for all $i, j \in I$. Prove that the union $U = \bigcup_{i\in I} U_i$ is connected.

Problem 4 (20%). Suppose for every $n \in \mathbb{N}$ that V_n is a non-empty closed subset of a compact space $\mathcal{X}$ with $V_n \supseteq V_{n+1}$. Prove

$$\bigcap_{n=1}^{+\infty} V_n \neq \varnothing.$$

Proof compact ⇒ seq. compact.

Problem 5 (20%). Let $\mathcal{X}$ be a complete metric space and $f\colon \mathcal{X} \to \mathcal{X}$ be a mapping such that for some $r > 1$ its r-time iteration $(f\circ)^r = f \circ \ldots \circ f$ is a Banach contraction.
Prove that f has a unique fixed point p in $\mathcal{X}$, regardless of f itself being (or maybe not?) a contraction.

Date: November 3, 2025. Good luck !